Fall 2011 Topology comprehensive exam

Akis, Beer, Krebs*

Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade—if you do more than five problems, we will select five to grade, and they may not be the five that you want us to grade.

Let $\mathbb{R}$ denote the set of real numbers. Unless otherwise stated, assume that $\mathbb{R}^n$ is endowed with the usual topology, and that subsets of a given topological space are endowed with the subspace topology.

(1) (a) Define what it means for two topological spaces to be homeomorphic.

For parts (b), (c), and (d), a topological space will be given. First determine if it is homeomorphic to the line $\mathbb{R}$, and then either produce a homeomorphism (but don't prove that it's a homeomorphism) between that that space and $\mathbb{R}$, or else explain why no such homeomorphism exists.

(b) The open interval $(0,1) = \{t \in \mathbb{R} : 0 < t < 1\}$
(c) The plane $\mathbb{R}^2$
(d) The circle $S = \{(x,y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$

(2) A subset A of the real line is declared open in the Sorgenfrey topology μ if $\forall a \in A$, there exists $\varepsilon > 0$ such that $[a, a+\varepsilon) \subseteq A$.
(a) Prove that the Sorgenfrey topology μ is finer than the usual topology τ.
(b) Give an example of a subset E which belongs to μ but does not belong to τ.
(c) Show that $[0,1]$ with the relative Sorgenfrey topology fails to be compact by producing an open cover with no finite subcover.

(3) Prove a metric space is separable if and only if it is second countable.

(4) (a) Prove that each decreasing sequence of nonempty closed sets $\langle A_n \rangle$ in a compact space $\langle X, \tau \rangle$ has nonempty intersection. (Recall that the sequence $\langle A_n \rangle$ is *decreasing* if $A_{j+1} \subset A_j$ for all j.)
(b) Produce a decreasing sequence of closed subsets of the Euclidean plane, each with nonempty interior, that has empty intersection.

(5) Suppose X is a topological space such that every continuous function $f : X \to X$ has a fixed point. Prove that X is connected. (Recall that a *fixed point* of a function f is a point x such that $f(x) = x$.)

(6) Suppose A, B are subsets of a topological space $\langle X, \tau \rangle$. Prove or give a counterexample to each statement below. Here we are using the notations $\mathrm{cl}(S)$ to denote the closure of a set S and $\mathrm{int}(S)$ to denote the interior of a set S.
(a) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$
(b) $\mathrm{int}(A \cup B) = \mathrm{int}(A) \cup \mathrm{int}(B)$
(c) If A is open, then $\mathrm{int}(\mathrm{cl}(A)) = A$.

(7) We say that a topological space X has *dimension zero* if there exists a basis for the topology on X where every basic open set is also closed in X. Let $\mathbb{Q}$ be the set of rational numbers. Endow $\mathbb{Q}$ with the subspace topology, as a subset of the set of real numbers with the usual topology. Prove that $\mathbb{Q}$ has dimension zero.