Fall 2012 Topology Comprehensive Exam

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Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade; otherwise, if you do more than five problems, we will select five to grade, and they may not be the five that you want us to grade.

1. (a) Prove that in a Hausdorff space $\langle X, \tau \rangle$, each compact set A is a closed subset of X.

   (b) Suppose $\mathscr{C}$ is a nonempty collection of compact subsets in a Hausdorff space $\langle X, \tau \rangle$. Prove that $\cap \mathscr{C}$ is compact.

2. Let $\langle X, \rho \rangle$ and $\langle Y, \mu \rangle$ be metric spaces. There are three standard metrics on $X \times Y$ that are compatible with the product topology:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{\rho(x_1, x_2)^2 + \mu(y_1, y_2)^2};$$
$$d((x_1, y_1), (x_2, y_2)) = \max\ \{\rho(x_1, x_2), \mu(y_1, y_2)\};$$
$$d((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \mu(y_1, y_2).$$

   Choose one of these, verify that it is a metric, and show that the metric topology determined by the one you chose agrees with the product topology on $X \times Y$.

3. Let A be a connected subset of $\langle X, \tau \rangle$ and let $\mathscr{C}$ be a collection of connected subsets such that $\forall C \in \mathscr{C},\ A \cap C \neq \emptyset$. Prove that $\cup \{C \cup A : C \in \mathscr{C}\}$ is connected.

4. Let $\overline{A}$ denote the closure of a subset A of a topological space $\langle X, \tau \rangle$.

   (a) Let $A \subseteq X$ and $B \subseteq X$; prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (b) Suppose $\langle Y, \sigma \rangle$ is also a topological space and $f : X \to Y$ is continuous. Show that if $A \subseteq X$, then $f(\overline{A}) \subseteq \overline{f(A)}$.

5. For any polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$, put $U_f := \{(x, y) \in \mathbb{R}^2 : f(x, y) \neq 0\}$. Prove that the collection $\{U_f : f : \mathbb{R}^2 \to \mathbb{R} \text{ is a polynomial}\}$ forms a basis for a topology on $\mathbb{R}^2$.

6. Recall that a topological space $\langle X, \tau \rangle$ is called *Lindelöf* if each open cover of X has a countable subcover.

   (a) Suppose $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ are topological spaces with $\langle X, \tau \rangle$ Lindelöf, and $f : X \to Y$ is continuous and onto. Show $\langle Y, \sigma \rangle$ is Lindelöf as well.

   (b) Suppose $\langle X, \tau \rangle$ is a topological space where X is a countable infinite set. Show X is Lindelöf.

7. (a) Let $\langle X, \tau \rangle$ be a normal Hausdorff space. Prove the space is completely regular.

   (b) Explain why any nonempty set X equipped with the discrete topology τ_d is normal.