**Fall 2016 Topology Comprehensive Exam**

Beer*, Krebs

Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade; otherwise, if you do more than five problems, we will grade the first five problems that you do. In the sequel $\mathbb{R}$ denotes the real numbers and $\mathbb{N}$ denotes the positive integers.

1. Recall that a set $A$ is called *countable* if it is empty or is nonempty and finite or it is nonempty and in one-to-one correspondence with $\mathbb{N}$. Let $\tau$ be the collection of subsets of $\mathbb{R}$ of the form $U\backslash T$ where $U$ is an open subset of $\mathbb{R}$ in the usual topology and $T$ is countable. Prove that $\tau$ is a topology which is strictly finer than the usual topology. Feel free to use standard facts about countable sets in your proof.

2. Let $\langle X, \tau\rangle$ be a topological space and suppose $f : X \to (0, \infty)$ is a function that is continuous at $p \in X$. Prove that $g : X \to (0, \infty)$ defined by $g(x) = \frac{1}{f(x)}$ is also continuous at $p$.

3. Let $\langle X, \tau\rangle$ and $\langle Y, \sigma\rangle$ be connected topological spaces. Prove that $X \times Y$ equipped with the product topology is connected.

4. Let $\sim$ be an equivalence relation on a nonempty set $X$. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and } x \sim y \\ 3 & \text{if } x \nsim y. \end{cases}$$

   (a) Prove that $d$ is a metric on $X$.

   (b) Describe the open balls of radius 2 in this metric space in terms of the equivalence relation.

5. Let $\langle X, d\rangle$ be a metric space and let $\tau_d$ be the topology determined by $d$. Prove that $\langle X, \tau_d\rangle$ is normal.

6. Let $\langle X, \tau\rangle$ be a Hausdorff space.

   (a) Prove that each $\tau$-compact subset of $X$ is closed.

   (b) Let $\sigma$ be the collection of subsets $A$ of $X$ such that either $A = \emptyset$ or $X\backslash A$ is $\tau$-compact. Prove using in part the assertion of (a) above that $\sigma$ is a topology on $X$.

7. Suppose $\langle X, \tau\rangle$ is a topological space with at least two points. Prove the following conditions are equivalent.

   (1) $\tau$ is either the discrete topology or the indiscrete topology;

   (2) Each function $f : \langle X, \tau\rangle \to \langle X, \tau\rangle$ is continuous.

   (*Hint*: for (2) $\Rightarrow$ (1), first show that if $A$ is a nonempty proper subset of $X$ and $p \in X$, $\exists f : X \to X$ with $f^{-1}(A) = \{p\}$)