Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of real numbers is denoted by R and the Euclidean plane is denoted by R^2. All spaces are assumed to be Hausdorff.

1. A space is called sequentially compact if each sequence in the space has a convergent subsequence. Prove that $X \times Y$ is sequentially compact if and only if the spaces X and Y are sequentially compact.

2. Let X and Y be spaces, and let $f : X \to Y$ be continuous and onto. Prove or disprove each of the following statements.

(a) If X is compact, then Y is compact.

(b) If Y is compact, then X is compact.

3. Prove that the following statements are equivalent for a space X:

(a) X is not connected;

(b) There exists a continuous onto function $f : X \to \{0, 1\}$ equipped with the discrete topology;

(c) There exists a non empty, proper subset A of X such that $\mathrm{bd}(A) = \emptyset$.

4. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ represent arbitrary points in R^2. Define metrics d and ρ on R^2 by

$$d(a, b) = |a_1 - b_1| + |a_2 - b_2|$$
$$\rho(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Prove these metrics are equivalent (you need not show they are metrics).

5. Let $f : R \to R$, and define $f_n : R \to R$ for by $f_n(x) = f(x + 1/n)$.

(a) Show that if f is continuous, then (f_n) converges pointwise to f.

(b) Show by an example that if f is continuous, then the convergence need not be uniform.

6. A subset of a topological space is called *nowhere dense* if its closure has empty interior. A subset of a topological space is called *totally disconnected* if all of the connected components of its closure are points. Prove or disprove each statement.

(a) Each nowhere dense subset A of R^2 is totally disconnected;

(b) Each totally disconnected subset A of R^2 is nowhere dense.

7. Prove that each regular compact space X is normal.

8. Consider the usual topology on R. Either exhibit a homeomorphism between the following subspaces or prove that one does not exist.

(a) $A = [0, 1]$ and $B = [3, 5]$.

(b) $C = [0, 1)$ and $D = [0, \infty)$.

If you exhibit a homeomorphism, you need not justify that it is one.