Topology Comprehensive Exam Spring 2018

Akis, Beer*, Krebs

Do any 5 of the 7 problems below. Each is worth 20 points. Please indicate clearly which 5 you want us to grade; otherwise we will grade your first 5 answers.

1. Let A and B be subsets of a topological space $\langle X, \tau \rangle$. One of the statements below is true and the other is false. Prove the correct statement and provide a counterexample for the incorrect statement.

   (a) $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ (b) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

2. Let p be a point of a metric space $\langle X, d \rangle$.

   (a) Prove that $x \mapsto d(x, p)$ is a continuous function on X.
   (b) Prove that $\{x \in X : 1 \leq d(x, p) \leq 2\}$ is a closed set.

3. Let $\langle X_1, \tau_1 \rangle$ and $\langle X_2, \tau_2 \rangle$ be topological spaces.

   (a) Carefully define the product topology τ_{prod} on $X_1 \times X_2$.
   (b) Let $\langle W, \sigma \rangle$ be a third topological space. Prove $f : W \to X_1 \times X_2$ is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous. Here, π_j is is the projection onto the jth coordinate space.

4. Equip the positive integers $\mathbb{N}$ with this topology τ: A is declared open iff either $1 \notin A$ or A contains all but finitely many elements of $\mathbb{N}$. Let $Y = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, equipped with the topology it inherits from $\mathbb{R}$ as a subspace. Prove that $\mathbb{N}$ and Y with these topologies are homeomorphic to each other.

5. Let $\sim$ be an equivalence relation on a nonempty set X. Define $d : X \times X \to \mathbb{R}$ by

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \sim b \text{ but } a \neq b \\ 3 & \text{otherwise.} \end{cases}$$

   Prove that d is a metric on X.

6. (a) Show that a topological space $\langle X, \tau \rangle$ is connected iff every continuous function on X into a discrete space (that is, a set equipped with the discrete topology) is constant.

   (b) Suppose $\langle X, \tau \rangle$ has finitely many connected components $\{C_1, C_2, \ldots, C_n\}$. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = k$ if $x \in C_k$ for each $k = 1, 2, \ldots, n$. Show f is continuous.

7. Recall that p is a *limit point* of a subset A of a topological space provided each neighborhood of p contains a point of A other than p.

   (a) Using the the fact that a compact subset of a Hausdorff space is closed, prove that if A is a compact subset of a Hausdorff space $\langle X, \tau \rangle$, then its set of limit points A' is compact.

   (b) Give an example with justification of a compact subset A of some (non-Hausdorff) topological space for which A' fails to be compact. (Suggestion: look for such a set in $[0, 1)$ equipped with the topology $\{\emptyset\} \cup \{[0, t) : 0 < t \leq 1\}$).