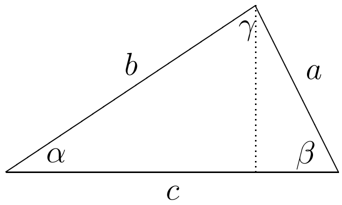
A Triangle Problem

Problem: Consider a triangle with sides and angles labeled as below.



Dropping a perpendicular as shown we see that $c = b\cos\alpha + a\cos\beta$. By symmetry, we get a system of three simultaneous equations:

$$a = b\cos\gamma + c\cos\beta$$
$$b = a\cos\gamma + c\cos\alpha$$
$$c = b\cos\alpha + a\cos\beta$$

One would expect that given angles $\alpha, \beta, \gamma \in (0, \pi)$, this system would be solvable with $a, b, c > 0$ if and only if $\alpha + \beta + \gamma = \pi$. Prove this.

Solution: The system can be written in the form $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ where

$$A = \begin{bmatrix} -1 & \cos\gamma & \cos\beta \\ \cos\gamma & -1 & \cos\alpha \\ \cos\beta & \cos\alpha & -1 \end{bmatrix}$$

Thus, for a nontrivial solution to exist, A must have zero determinant:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1 = 0.$$

Rewrite this equation as a quadratic polynomial in $\cos\alpha$:

$$(\cos\alpha)^2 + [2\cos\beta\cos\gamma]\cos\alpha + [\cos^2\beta + \cos^2\gamma - 1] = 0.$$

The discriminant is

$$\begin{aligned}(2\cos\beta\cos\gamma)^2 - 4(\cos^2\beta + \cos^2\gamma - 1) &= 4(1-\cos^2\beta)(1-\cos^2\gamma) \\ &= 4\sin^2\beta\sin^2\gamma,\end{aligned}$$

and so, from the quadratic formula,

$$\cos\alpha = -\cos\beta\cos\gamma \pm \sin\beta\sin\gamma = -\cos(\beta\pm\gamma) = \cos(\pi+\beta\pm\gamma).$$

Thus $\pm\alpha = \pi + \beta \pm \gamma + 2\pi n$ for some $n \in \mathbb{Z}$.
Since $\alpha, \beta, \gamma \in (0, \pi)$, we have, in fact, the following cases:

1) $\alpha + \beta + \gamma = \pi$
2a) $\alpha + \beta - \gamma = \pi$
2b) $\alpha - \beta + \gamma = \pi$
2c) $-\alpha + \beta + \gamma = \pi$

In case 1), $\sin\alpha = \sin(\pi - \beta - \gamma) = \sin(\beta + \gamma) = \sin\beta\cos\gamma + \cos\beta\sin\gamma$, and similarly, $\sin\beta = \sin\alpha\cos\gamma + \cos\alpha\sin\gamma$, $\sin\gamma = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Since $\sin\alpha, \sin\beta, \sin\gamma > 0$, $a = \sin\alpha$, $b = \sin\beta$ and $c = \sin\gamma$ is a solution of the problem.

In case 2a), replacing γ by $-\gamma$ in the above shows that $\begin{bmatrix} \sin\alpha \\ \sin\beta \\ -\sin\gamma \end{bmatrix}$ is in the null space of A. In fact, this column vector is a basis for the null space of A since the rank of A is 2. For example,

$$\det \begin{bmatrix} -1 & \cos\gamma \\ \cos\gamma & -1 \end{bmatrix} = \sin^2\gamma > 0$$

Since no multiple of this column vector has all positive entries, we get no solutions of the original problem in this case.
By symmetry, cases 2b) and 2c) yield no further solutions.