Midterm, Math 209, Winter 2007 **Name:** Solutions

Directions: Show ALL of your work to get credit. If you leave something out, then you may be penalized. No calculators. Good luck!

1. [10 points] Find the volume of the solid that lies under the hyperbolic paraboloid

$$z = 4 + x^2 - y^2$$

and above the region

$$R = [-1, 1] \times [0, 2] = \{(x, y) \mid -1 \leq x \leq 1 \, , \, 0 \leq y \leq 2\}.$$

$$\int_{-1}^{1}\int_{0}^{2} (4+x^2-y^2)\,dy\,dx = \int_{-1}^{1} \left(4y + x^2 y - \frac{y^3}{3}\right)\Big|_0^2 dx$$

$$= \int_{-1}^{1} \left(\frac{16}{3} + 2x^2\right) dx = \left(\frac{16}{3}x + \frac{2x^3}{3}\right)\Big|_{-1}^{1} = \left(\frac{16}{3} + \frac{2}{3}\right) - \left(-\frac{16}{3} - \frac{2}{3}\right)$$

$$= \frac{36}{3} = 12$$

2. [10 points] Use the definition of partial derivatives as limits to find $f_y(x, y)$ where

$$f(x, y) = 2x^2 + y^2 + 5y.$$

$$\lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h \to 0} \frac{2x^2 + (y+h)^2 + 5(y+h) - [2x^2 + y^2 + 5y]}{h}$$

$$= \lim_{h \to 0} \frac{2yh + h^2 + 5h}{h} = \lim_{h \to 0} (2y + h + 5) = 2y + 5$$

3. [10 points] Let

$$z = x^2y^2 - x + 2y, \quad x = \sqrt{s}, \quad y = t^3 e^{s^2+t^2-2}$$

Compute $\frac{\partial z}{\partial s}$ at the point $s = 1$ and $t = 1$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2xy^2 - 1)\left(\tfrac{1}{2} s^{-1/2}\right) + (2x^2y + 2)\left(t^3 \cdot (2s) \cdot e^{s^2+t^2-2}\right)$$

$$= (1)\left(\tfrac{1}{2}\right) + (4)(2) = \tfrac{1}{2} + 8 = \frac{17}{2}$$

$\uparrow$

$x = \sqrt{1} = 1$

$y = e^0 = 1$

4. [10 points] Consider the function $f(x, y) = \sqrt{x} + y$.

(a) Find $L(x, y)$ at the point $(1, 1)$.

(b) Use your answer from part (a) to approximate $f(1.1, 1)$. [Recall that $\frac{1}{2} = 0.5$]

(a) $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$$= 2 + \tfrac{1}{2}(x-1) + (y-1) = \tfrac{1}{2} + \tfrac{1}{2}x + y$$

$\uparrow$

$f_x = \frac{1}{2}\frac{1}{\sqrt{x}}$

$f_y = 1$

(b) $f(1.1, 1) \approx L(1.1, 1) = \frac{1}{2} + \frac{1}{2}(1.1) + 1$

$$= 0.5 + 0.5(1.1) + 1$$

2

$$= 0.5 + 0.55 + 1 = 2.05$$

$$\begin{array}{r} 1.1 \\ 0.5 \\ \hline .55 \end{array}$$

5. [10 points] Consider the function $f(x, y) = \ln(x^2 + y^2)$.

(a) Find the directional derivative of $f(x, y)$ at $(2, 1)$ in the direction of $v = < -1, 2 >$.

(b) Find the maximum rate of change of $f(x, y)$ at $(2, 1)$, and the direction in which it occurs.

(a) $\vec{v} = \langle -1, 2 \rangle$

$\|\vec{v}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$\vec{u} = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \langle \frac{4}{5}, \frac{2}{5} \rangle \cdot \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle =$

$\uparrow$

$f_x = \frac{2x}{x^2 + y^2}$

$f_y = \frac{2y}{x^2 + y^2}$

$= \frac{-4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$

(b) maximum rate of change: $|\nabla f(2,1)| = |\langle \frac{4}{5}, \frac{2}{5} \rangle|$

$= \sqrt{(\frac{4}{5})^2 + (\frac{2}{5})^2}$

$= \frac{\sqrt{20}}{5}$

direction: $\nabla f(2,1) = \langle \frac{4}{5}, \frac{2}{5} \rangle$

6. [10 points] Find the maximum and minimum values of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 25$.

$\nabla f(x,y) = \langle 2x, -2y \rangle$

$\lambda\nabla g(x,y) = \lambda\langle 2x, 2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$

$$\left.\begin{array}{l} ① \; 2x = 2\lambda x \\ ② \; -2y = 2\lambda y \\ ③ \; x^2 + y^2 = 25 \end{array}\right\}$$

Consider equation ①. Either $x = 0$ or $\lambda = \frac{2x}{2x} = 1$.

If $x = 0$, then equation ③ gives $y^2 = 25$. So, $y = \pm 5$.

This works in equation ② since it becomes $\lambda = -1$.

So, we get the solutions: $(0, 5)$, $(0, -5)$

If $x \neq 0$, and $\lambda = 1$, then equation ② becomes

$-2y = 2y \Rightarrow y = 0$. Then, by ③ $x^2 = 25 \Rightarrow x = \pm 5$.

So, we get the points: $(5, 0)$, $(-5, 0)$.

Checking for abs max/min:

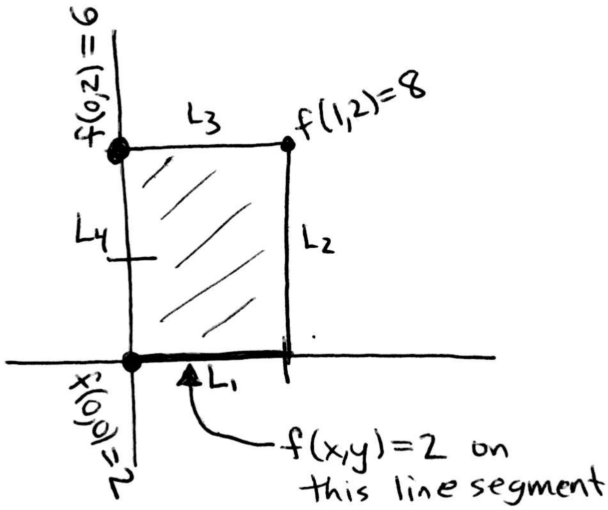
$$\left.\begin{array}{l} f(0,5) = -25 \\ f(0,-5) = -25 \end{array}\right\} \text{absolute mins}$$

$$\left.\begin{array}{l} f(5,0) = 25 \\ f(-5,0) = 25 \end{array}\right\} \text{absolute maxs}$$

7. [10 points] Find the absolute maximum and minimum values of

$$f(x, y) = y^2 + xy + 2$$

on the rectangle given by the set $D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2\}$.



1. Find critical points:

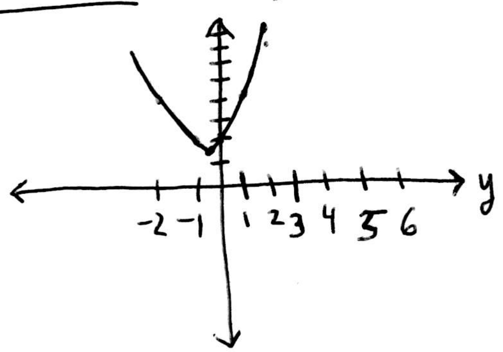
$$\left.\begin{array}{l} f_x(x,y) = y = 0 \\ f_y(x,y) = 2y + x = 0 \end{array}\right\} \begin{array}{l} y = 0 \\ x = 0 \end{array}$$

$(0,0)$ is the critical point.

2. Check boundary:

<u>On L_1:</u> $(y=0,\ 0 \leq x \leq 1)$, $f(x,0) = 2$.

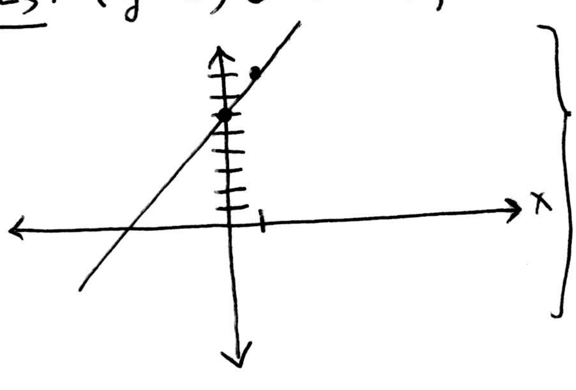
<u>On L_2:</u> $(x=1,\ 0 \leq y \leq 2)$, $f(1,y) = y^2 + y + 2$



minimum of y^2+y+2 occurs when derivative is 0.
$2y+1=0 \Rightarrow y = -\frac{1}{2}$

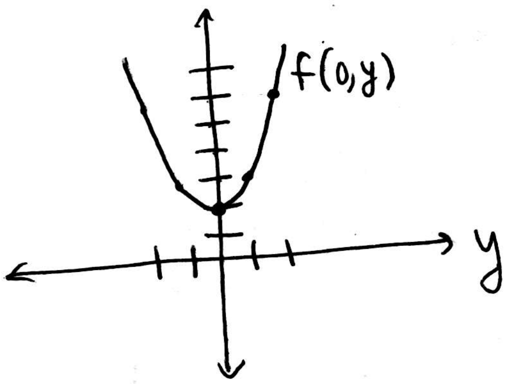
So, maximum on L_2 occurs at $(x,y) = (1,2)$ and minimum at $(x,y) = (0,2)$

<u>On L_3:</u> $(y=2,\ 0 \leq x \leq 1)$, $f(x,2) = 6 + 2x$



min at $(x,y) = (0,2)$
max at $(x,y) = (1,2)$

<u>On L_4:</u> $(x=0,\ 0 \le y \le 2)$, $f(0,y) = y^2+2$



max at $(x,y)=(0,2)$
min at $(x,y)=(0,0)$

See picture on other page of this problem.

Maximum of $f(x,y)$ occurs when $(x,y)=(1,2)$.

Minumum occurs on the line L_1; that is, when $y=0$ and $0 \le x \le 1$.

Max value is 8, min value is 2.